



5.3 n 维向量空间的正交化

主要内容: 内积

标准正交基

施密特正交化方法

正交矩阵



一. 内积

1. 定义: 设 $\alpha = (a_1, a_2, \dots, a_n)$, $\beta = (b_1, b_2, \dots, b_n)$

$$(\alpha, \beta) = a_1 b_1 + a_2 b_2 + \dots + a_n b_n$$

称为 α 与 β 的内积.

2. 性质:

(1) $(\alpha, \beta) = (\beta, \alpha)$;

(2) $(\alpha + \beta, \gamma) = (\alpha, \gamma) + (\beta, \gamma)$

$(k\alpha, \beta) = k(\alpha, \beta)$;

(3) $(\alpha, \alpha) \geq 0$, 当且仅当 $\alpha = 0$ 时等号成立.

内积还满足以下关系：

$$(\alpha, l\beta) = l(\alpha, \beta), \quad l \in \mathbb{R};$$

$$(\alpha, \beta + \gamma) = (\alpha, \beta) + (\alpha, \gamma).$$

3. 长度

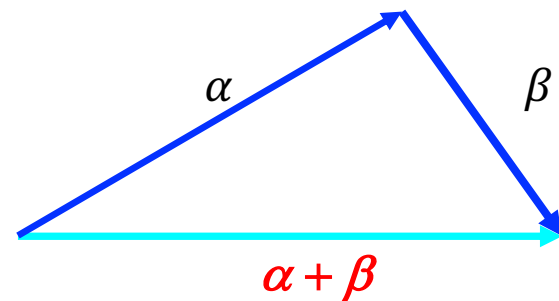
(1) 定义 $\|\alpha\| = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2} = \sqrt{(\alpha, \alpha)}$

(2) 性质

1° 非负性 $\|\alpha\| \geq 0$;

2° 齐次性 $\|k\alpha\| = |k| \|\alpha\|$;

3° 三角不等式 $\|\alpha + \beta\| \leq \|\alpha\| + \|\beta\|$.



(3) 单位向量

$\|\alpha\| = 1$: α 为单位向量.

设 $\alpha \neq 0$, 令 $\alpha_e = \frac{1}{\|\alpha\|} \alpha$, 则

$$\|\alpha_e\| = \sqrt{(\alpha_e, \alpha_e)} = \sqrt{\frac{1}{\|\alpha\|^2} (\alpha, \alpha)} = 1.$$

4. 夹角

$\langle \alpha, \beta \rangle = \arccos \frac{(\alpha, \beta)}{\|\alpha\| \|\beta\|}$: α 与 β 的夹角.

问题: $\left| \frac{(\alpha, \beta)}{\|\alpha\| \|\beta\|} \right| \leq 1$?

柯西不等式 $|(\alpha, \beta)| \leq \|\alpha\| \|\beta\|$,

当且仅当 α 与 β 线性相关时等号成立。

证 (1) α, β 线性无关: $\forall t \in \mathbb{R}, t\alpha + \beta \neq 0$,

$$(t\alpha + \beta, t\alpha + \beta) = t^2(\alpha, \alpha) + 2t(\alpha, \beta) + (\beta, \beta) > 0,$$

$$\therefore [2(\alpha, \beta)]^2 - 4(\alpha, \alpha)(\beta, \beta) < 0,$$

$$(\alpha, \beta)^2 < \|\alpha\|^2 \|\beta\|^2, \quad |(\alpha, \beta)| < \|\alpha\| \|\beta\|.$$

(2) α, β 线性相关: 设 $\beta = k\alpha$, 则

$$(\alpha, \beta)^2 = (\alpha, k\alpha)^2 = k^2(\alpha, \alpha)^2 = (\alpha, \alpha)(k\alpha, k\alpha) = \|\alpha\|^2 \|\beta\|^2,$$

$$|(\alpha, \beta)| = \|\alpha\| \|\beta\|.$$



二. 规范正交基

1. 正交向量组

α 与 β 正交: $(\alpha, \beta) = 0$.

$\alpha_1, \alpha_2, \dots, \alpha_s$ 为正交向量组:

两两正交且不含零向量 .

如: $\alpha_1 = (1, 1, 1), \alpha_2 = (-1, 2, -1), \alpha_3 = (-1, 0, 1)$

$$(\alpha_1, \alpha_2) = (\alpha_1, \alpha_3) = (\alpha_2, \alpha_3) = 0$$

$\alpha_1, \alpha_2, \alpha_3$ 为正交向量组 .



例1 设 A 是 n 阶反对称矩阵, x 是 n 维列向量, 且 $Ax=y$, 证明: x 与 y 正交.

$$\text{证: } (x, y) = x^T y = x^T Ax$$

$$(y, x) = y^T x = (Ax)^T x = x^T A^T x = -x^T Ax,$$

由 $(x, y) = (y, x)$ 可知:

$$(x, y) = 0.$$



定理1 正交向量组线性无关.

证 设 $\alpha_1, \alpha_2, \dots, \alpha_s$ 为正交向量组, 且

$$k_1\alpha_1 + k_2\alpha_2 + \dots + k_s\alpha_s = \mathbf{0}$$

则 $(\alpha_1, k_1\alpha_1 + k_2\alpha_2 + \dots + k_s\alpha_s)$

$$= k_1(\alpha_1, \alpha_1) + k_2(\alpha_1, \alpha_2) + \dots + k_s(\alpha_1, \alpha_s)$$

$$= k_1(\alpha_1, \alpha_1) = 0,$$

$$\because (\alpha_1, \alpha_1) > 0, \quad \therefore k_1 = 0,$$

$$\text{同理: } k_2 = k_3 = \dots = k_s = 0,$$

$\therefore \alpha_1, \alpha_2, \dots, \alpha_s$ 线性无关.



线性无关向量组未必是正交向量组 .

如: $\alpha_1 = (1, 0, 0)$, $\alpha_2 = (1, 1, 0)$, $\alpha_3 = (1, 1, 1)$

例2 $\alpha_1 = (1, 1, 1)$, $\alpha_2 = (1, -2, 1)$,

求 α_3 , 使 $\alpha_1, \alpha_2, \alpha_3$ 为正交向量组.

解 设 $\alpha_3 = (x_1, x_2, x_3)$, 则

$$(\alpha_1, \alpha_3) = x_1 + x_2 + x_3 = 0$$

$$(\alpha_2, \alpha_3) = x_1 - 2x_2 + x_3 = 0$$

$$\alpha_3 = (1, 0, -1).$$



2. 规范正交向量组

$\alpha_1, \alpha_2, \dots, \alpha_s$ 满足 :

$$(1) (\alpha_i, \alpha_j) = 0, (i \neq j, \alpha_i \neq 0, \alpha_j \neq 0)$$

$$(2) \|\alpha_i\| = 1, (i = 1, 2, \dots, s)$$

则称 $\alpha_1, \alpha_2, \dots, \alpha_s$ 为规范(标准)正交向量组 .

如 $\varepsilon_1 = (1, 0, \dots, 0), \varepsilon_2 = (0, 1, \dots, 0), \dots, \varepsilon_n = (0, 0, \dots, 1)$
是 R^n 的规范正交基 .

$$\alpha_1 = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right), \alpha_2 = \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right), \alpha_3 = (0, 1, 0)$$

是 R^3 的规范正交基 .



三. 施密特正交化方法

任一线性无关向量组都可规范正交化 .

设 $\alpha_1, \alpha_2, \alpha_3$ 线性无关, 确定与 $\alpha_1, \alpha_2, \alpha_3$ 等价的正交向量组 $\beta_1, \beta_2, \beta_3$.

令 $\beta_1 = \alpha_1, \beta_2 = \alpha_2 + k\beta_1$, 选择适当的 k , 使 $(\beta_1, \beta_2) = 0$, 即

$$(\alpha_2 + k\beta_1, \beta_1) = (\alpha_2, \beta_1) + k(\beta_1, \beta_1) = 0,$$

$$k = -\frac{(\alpha_2, \beta_1)}{(\beta_1, \beta_1)}, \quad \beta_2 = \alpha_2 - \frac{(\alpha_2, \beta_1)}{(\beta_1, \beta_1)}\beta_1.$$



令 $\beta_3 = \alpha_3 + k_1\beta_1 + k_2\beta_2$, 为使

$(\beta_1, \beta_3) = (\beta_2, \beta_3) = 0$, 则可推出

$$k_1 = -\frac{(\alpha_3, \beta_1)}{(\beta_1, \beta_1)}, \quad k_2 = -\frac{(\alpha_3, \beta_2)}{(\beta_2, \beta_2)},$$

于是

$$\beta_3 = \alpha_3 - \frac{(\alpha_3, \beta_1)}{(\beta_1, \beta_1)}\beta_1 - \frac{(\alpha_3, \beta_2)}{(\beta_2, \beta_2)}\beta_2,$$

$\beta_1, \beta_2, \beta_3$ 是与 $\alpha_1, \alpha_2, \alpha_3$ 等价的正交向量组 .



把线性无关向量组 $\alpha_1, \alpha_2, \dots, \alpha_s$ 规范正交化

$$\beta_1 = \alpha_1$$

$$\beta_2 = \alpha_2 - \frac{(\alpha_2, \beta_1)}{(\beta_1, \beta_1)} \beta_1$$

$$\beta_3 = \alpha_3 - \frac{(\alpha_3, \beta_1)}{(\beta_1, \beta_1)} \beta_1 - \frac{(\alpha_3, \beta_2)}{(\beta_2, \beta_2)} \beta_2$$

... ..

$$\beta_s = \alpha_s - \frac{(\alpha_s, \beta_1)}{(\beta_1, \beta_1)} \beta_1 - \frac{(\alpha_s, \beta_2)}{(\beta_2, \beta_2)} \beta_2 - \dots - \frac{(\alpha_s, \beta_{s-1})}{(\beta_{s-1}, \beta_{s-1})} \beta_{s-1}.$$

再令 $\gamma_i = \frac{1}{\|\beta_i\|} \beta_i$ ($i = 1, 2, \dots, s$), $\gamma_1, \gamma_2, \dots, \gamma_s$ 为规范正交向量组 .



施密特正交化方法

例3 设 $\alpha_1 = (1, 1, 1)$, 求 α_2, α_3 , 使 $\alpha_1, \alpha_2, \alpha_3$ 为正交向量组.

解 设与 α_1 正交的向量为 $\alpha = (x_1, x_2, x_3)$, 则

$$(\alpha_1, \alpha) = x_1 + x_2 + x_3 = 0$$

其基础解系为

$$X_1 = (1, 0, -1), \quad X_2 = (0, 1, -1)$$

将 X_1, X_2 正交化:

$$\alpha_2 = X_1 = (1, 0, -1),$$

$$\alpha_3 = X_2 - \frac{(X_2, \alpha_2)}{(\alpha_2, \alpha_2)} \alpha_2 = (0, 1, -1) - \frac{1}{2}(1, 0, -1)$$

$$= \frac{1}{2}(-1, 2, -1).$$



例4 将 $\alpha_1 = (1, 1, 1)$, $\alpha_2 = (1, 2, 1)$, $\alpha_3 = (0, -1, 1)$
规范正交化.

$$\begin{aligned}\text{解 设 } \beta_1 &= \alpha_1 = (1, 1, 1), \\ \beta_2 &= \alpha_2 - \frac{(\alpha_2, \beta_1)}{(\beta_1, \beta_1)} \beta_1 = (1, 2, 1) - \frac{4}{3}(1, 1, 1) \\ &= \frac{1}{3}(-1, 2, -1), \\ \beta_3 &= \alpha_3 - \frac{(\alpha_3, \beta_1)}{(\beta_1, \beta_1)} \beta_1 - \frac{(\alpha_3, \beta_2)}{(\beta_2, \beta_2)} \beta_2 \\ &= \dots = \frac{1}{2}(-1, 0, 1),\end{aligned}$$



施密特正交化方法

$$\gamma_1 = \frac{1}{\|\beta_1\|} \beta_1 = \frac{1}{\sqrt{3}}(1, 1, 1)$$

$$\gamma_2 = \frac{1}{\|\beta_2\|} \beta_2 = \frac{1}{\sqrt{6}}(-1, 2, -1)$$

$$\gamma_3 = \frac{1}{\|\beta_3\|} \beta_3 = \frac{1}{\sqrt{2}}(-1, 0, 1).$$

注意：将 $\beta = \frac{1}{k}\alpha$ 单位化，只需将 α 单位化即可。

为什么？

$$\gamma = \frac{1}{\|\beta\|} \beta = \frac{1}{\sqrt{\left(\frac{1}{k}\alpha, \frac{1}{k}\alpha\right)}} \frac{1}{k}\alpha = \frac{k}{|\alpha|} \frac{1}{k}\alpha = \pm \frac{1}{|\alpha|} \alpha.$$



四. 正交矩阵

将例4中的 $\gamma_1, \gamma_2, \gamma_3$ 作矩阵 A 的列向量组:

$$A = (\gamma_1 \quad \gamma_2 \quad \gamma_3) = \begin{pmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \\ 1 & 2 & 0 \\ \sqrt{3} & \sqrt{6} & 0 \\ 1 & -1 & 1 \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$AA^T = \begin{pmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \\ 1 & 2 & 0 \\ \sqrt{3} & \sqrt{6} & 0 \\ 1 & -1 & 1 \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 1 & 2 & 1 \\ -\sqrt{6} & \sqrt{6} & -\sqrt{6} \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} = I$$



1. 定义 若实矩阵 A 满足 $AA^T=A^T A=I$, 则称 A 正交矩阵 .

2. 性质 (1) $A^{-1} = A^T$,

(2) $|A| = \pm 1$,

$$|A^T A| = |A^T| |A| = |A|^2 = |I| = 1 .$$

(3) 正交矩阵的乘积也是正交矩阵 .

设 $A^T A = AA^T = I$ $B^T B = BB^T = I$, 则

$$(AB)^T (AB) = B^T A^T AB = B^T B = I .$$

(4) A 为正交矩阵 $\Leftrightarrow A$ 的行(列)向量组都是规范正交向量组 .



证 设 $A = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}$ $A^T = (\alpha_1^T \quad \alpha_2^T \quad \cdots \quad \alpha_n^T)$, 则

$$AA^T = \begin{pmatrix} \alpha_1\alpha_1^T & \alpha_1\alpha_2^T & \cdots & \alpha_1\alpha_n^T \\ \alpha_2\alpha_1^T & \alpha_2\alpha_2^T & \cdots & \alpha_2\alpha_n^T \\ \cdots & \cdots & \cdots & \cdots \\ \alpha_n\alpha_1^T & \alpha_n\alpha_2^T & \cdots & \alpha_n\alpha_n^T \end{pmatrix} = I$$

$$\Leftrightarrow \alpha_i\alpha_i^T = 1, \quad \alpha_i\alpha_j^T = 0 (i \neq j). \quad \Leftrightarrow (\alpha_i, \alpha_i) = 1, \quad (\alpha_i, \alpha_j) = 0 (i \neq j).$$

例5 设 $\alpha_1, \alpha_2, \alpha_3$ 都是3维实列向量，且

$A = (\alpha_1 \ \alpha_2 \ \alpha_3)$ 为正交矩阵，

$$\beta_1 = \frac{1}{3}(2\alpha_1 + 2\alpha_2 - \alpha_3),$$

$$\beta_2 = \frac{1}{3}(2\alpha_1 - \alpha_2 + 2\alpha_3),$$

$$\beta_3 = \frac{1}{3}(\alpha_1 - 2\alpha_2 - 2\alpha_3),$$

证明： $B = (\beta_1 \ \beta_2 \ \beta_3)$ 是正交矩阵。

分析：只需证明

$$(\beta_i, \beta_j) = 0 \ (i \neq j), \quad \|\beta_i\| = 1, \ (i = 1, 2, 3).$$



证 $\because A = (\alpha_1 \alpha_2 \alpha_3)$ 为正交矩阵,

$$\therefore (\alpha_i, \alpha_j) = 0 \quad (i \neq j), \quad (\alpha_i, \alpha_i) = 1 \quad (i = 1, 2, 3).$$

$$\begin{aligned} (\beta_1, \beta_2) &= \left(\frac{2}{3}\alpha_1 + \frac{2}{3}\alpha_2 - \frac{1}{3}\alpha_3, \frac{2}{3}\alpha_1 - \frac{1}{3}\alpha_2 + \frac{2}{3}\alpha_3 \right) \\ &= \frac{4}{9}(\alpha_1, \alpha_1) - \frac{2}{9}(\alpha_2, \alpha_2) - \frac{2}{9}(\alpha_3, \alpha_3) = 0, \end{aligned}$$

同样, $(\beta_1, \beta_3) = (\beta_2, \beta_3) = 0$.

$$\begin{aligned} \|\beta_1\| &= \sqrt{(\beta_1, \beta_1)} \\ &= \sqrt{\frac{4}{9}(\alpha_1, \alpha_1) + \frac{4}{9}(\alpha_2, \alpha_2) + \frac{1}{9}(\alpha_3, \alpha_3)} = 1, \quad \text{同样, } \|\beta_2\| = \|\beta_3\| = 1. \end{aligned}$$



例6 设 A 是奇数阶正交矩阵且 $\det A=1$.

证明： 1 是 A 的特征值 .

分析：(1) 是否存在向量 α , 使 $A\alpha = 1\alpha$?

(2) $|1I - A| = 0$?

$$\begin{aligned} \text{证： } |1I - A| &= |AA^T - A| = |A| |A^T - I| = |(A - I)^T| \\ &= |A - I| = (-1)^n |I - A| = -|1I - A| \\ \therefore |1I - A| &= 0 . \end{aligned}$$



内积

标准正交基

施密特正交化方法

正交矩阵