

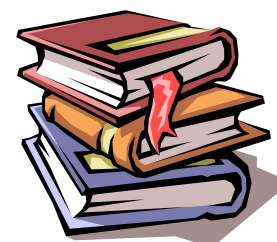


2.1 n 阶行列式的定义

主要内容:

一. 一、二、三阶行列式

二. n 阶行列式的定义





一、二、三阶行列式

一阶行列式: $|a_{11}| = a_{11}$ 如, 行列式 $|-5| = -5$, $|3| = 3$

二阶行列式:
$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

如,
$$\begin{vmatrix} 2 & -1 \\ 3 & -3 \end{vmatrix} = -6 + 3 = -3$$

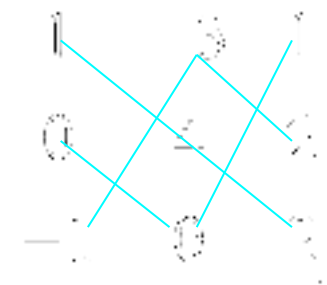
一、二、三阶行列式

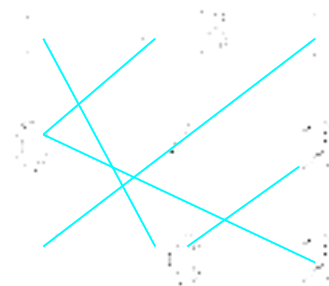
三阶行列式:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}$$

三阶行列式计算式的记忆法

例如

$$\begin{vmatrix} 1 & 3 & 1 \\ 0 & 4 & 2 \\ -1 & 0 & 2 \end{vmatrix} = 1 \cdot 4 \cdot 2 - 3 \cdot 1 \cdot 4 + 1 \cdot 0 \cdot 2 = 8 - 12 + 0 = -4$$






一、二、三阶行列式

例1 计算三阶行列式 $D = \begin{vmatrix} 1 & 2 & -4 \\ -2 & 2 & 1 \\ -3 & 4 & -2 \end{vmatrix}$

解 按对角线法则，有

$$D = 1 \times 2 \times (-2) + 2 \times 1 \times (-3) + (-4) \times (-2) \times 4 \\ - 1 \times 1 \times 4 - 2 \times (-2) \times (-2) - (-4) \times 2 \times (-3)$$



一、二、三阶行列式

例2 求解方程 $\begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & x \\ 4 & 9 & x^2 \end{vmatrix} = 0.$

解 方程左端

$$\begin{aligned} D &= 3x^2 + 4x + 18 - 9x - 2x^2 - 12 \\ &= x^2 - 5x + 6, \end{aligned}$$

由 $x^2 - 5x + 6 = 0$ 解得

$$x = 2 \text{ 或 } x = 3.$$

一、二、三阶行列式

二、三阶行列式计算式规律的观察：

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21} = a_{11}A_{11} + a_{12}A_{12}$$

$$A_{11} = (-1)^{1+1} |a_{22}|, \quad A_{12} = (-1)^{1+2} |a_{21}|$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{12} (-1) \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$$



一、二、三阶行列式

称：

$$A_{11} = (-1)^{1+1} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}, A_{12} = (-1)^{1+2} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

$$A_{13} = (-1)^{1+3} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}, \text{分别为 } a_{11}, a_{12}, a_{13}$$

的代数余子式

n阶行列式的定义

定义：定义n阶矩阵A的行列式 $\det A = \begin{vmatrix} \mathbf{a}_{11} & \mathbf{a}_{12} & \cdots & \mathbf{a}_{1n} \\ \mathbf{a}_{21} & \mathbf{a}_{22} & \cdots & \mathbf{a}_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \mathbf{a}_{n1} & \mathbf{a}_{n2} & \cdots & \mathbf{a}_{nn} \end{vmatrix}$

(1) 当 $n = 1$ 时, $\det A = \det(\mathbf{a}_{11}) = \mathbf{a}_{11}$;

(2) 当 $n \geq 2$ 时,

$$\det A = \mathbf{a}_{11}A_{11} + \mathbf{a}_{12}A_{12} + \cdots + \mathbf{a}_{1n}A_{1n},$$

其中 $A_{1j} = (-1)^{1+j} M_{1j}$, M_{1j} 为划去A的第1行第j列后所得的 $n - 1$ 阶行列式,

记号 $\det A, |A|$



行列式与矩阵的区别与联系：

(1) $D_{n \times n}$, $A_{m \times n}$;

(2) 数 , 数表 ;

(3) $| \quad |$, (\quad) , $[\quad]$;

(4) $A_{n \times n} \rightarrow |A| = \det A .$



n阶行列式的定义

例3 求 $\det A$:

$$A = \begin{pmatrix} 1 & -3 & 7 \\ 2 & 4 & -3 \\ -3 & 7 & 2 \end{pmatrix}$$

解 $\det A = 1(-1)^{1+1} \begin{vmatrix} 4 & -3 \\ 7 & 2 \end{vmatrix} + (-3)(-1)^{1+2} \begin{vmatrix} 2 & -3 \\ -3 & 2 \end{vmatrix} \\ + 7(-1)^{1+3} \begin{vmatrix} 2 & 4 \\ -3 & 7 \end{vmatrix}$

$$= (8 + 21) + 3(4 - 9) + 7(14 + 12) = 196$$

n阶行列式的定义

例4 计算

$$D_n = \begin{vmatrix} a_{11} & & & \\ a_{21} & a_{22} & 0 & \\ \vdots & \vdots & \ddots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

解

$$D_n = a_{11} \begin{vmatrix} a_{22} & & & \\ a_{32} & a_{33} & 0 & \\ \vdots & \vdots & \ddots & \\ a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix} = a_{11} a_{22} \begin{vmatrix} a_{33} & & & \\ a_{43} & a_{44} & 0 & \\ \vdots & \vdots & \ddots & \\ a_{n3} & a_{n4} & \cdots & a_{nn} \end{vmatrix}$$

$$= \cdots = a_{11} a_{22} \cdots a_{nn}$$



n阶行列式的定义

$$\text{同理, } \det(\text{diag}(a_{11}, a_{22}, \dots, a_{nn})) = a_{11}a_{22}\cdots a_{nn}$$

$$\det I = 1, \quad \det(kI_n) = k^n$$

例5 计算斜下三角行列式

$$D_n = \begin{vmatrix} & & & a_n \\ & & \ddots & \\ & & a_2 & \\ a_1 & & & * \end{vmatrix}$$

n阶行列式的定义

解:

$$D_n = a_n (-1)^{1+n} \begin{vmatrix} 0 & & a_{n-1} \\ & \ddots & \\ a_2 & & * \\ a_1 & & \end{vmatrix} = (-1)^{n-1} a_n D_{n-1}$$

$$= (-1)^{n-1} a_n (-1)^{n-2} D_{n-2}$$

$$= \cdots = (-1)^{(n-1)+(n-2)+\cdots+1} a_n a_{n-1} \cdots a_1$$

$$= (-1)^{\frac{n(n-1)}{2}} a_1 a_2 \cdots a_n$$

n阶行列式的定义

同理, $D_n = \begin{vmatrix} 0 & & a_n \\ & \ddots & \\ a_2 & & \\ a_1 & & 0 \end{vmatrix} = (-1)^{\frac{n(n-1)}{2}} a_1 a_2 \cdots a_n$



一. 一、二、三阶行列式

二. n 阶行列式的定义